

Quantum Lie algebras, their existence, uniqueness and q -antisymmetry

Gustav W. Delius

Department of Mathematics, King's College London
Strand, London WC2R 2LS, Great Britain
e-mail: delius@math.kcl.ac.uk
<http://www.math.kcl.ac.uk/~delius>

Mark D. Gould

Department of Mathematics, University of Queensland
Brisbane Qld 4072, Australia

Abstract

Quantum Lie algebras are generalizations of Lie algebras which have the quantum parameter h built into their structure. They have been defined concretely as certain submodules $\mathfrak{L}_h(\mathfrak{g})$ of the quantized enveloping algebras $U_h(\mathfrak{g})$. On them the quantum Lie product is given by the quantum adjoint action.

Here we define for any finite-dimensional simple complex Lie algebra \mathfrak{g} an abstract quantum Lie algebra \mathfrak{g}_h independent of any concrete realization. Its h -dependent structure constants are given in terms of inverse quantum Clebsch-Gordan coefficients. We then show that all concrete quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ are isomorphic to an abstract quantum Lie algebra \mathfrak{g}_h .

In this way we prove two important properties of quantum Lie algebras: 1) all quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ associated to the same \mathfrak{g} are isomorphic, 2) the quantum Lie product of any $\mathfrak{L}_h(\mathfrak{g})$ is q -antisymmetric. We also describe a construction of $\mathfrak{L}_h(\mathfrak{g})$ which establishes their existence.

1 Introduction

Lie algebras play an important role in the description of many classical physical theories. This is particularly pronounced in integrable models which are described entirely in terms of Lie algebraic data. However, when quantizing a classical theory the Lie algebraic description seems to be destroyed by quantum corrections.

It is conceivable that in some cases the Lie algebraic structure of the theory is deformed rather than destroyed. The quantum theory may be describable by a quantum generalization of a Lie algebra which has higher order terms in \hbar built into its structure. These speculations were prompted by the beautiful structure found in affine Toda quantum field theories [1]. They are the physical motivation for this work on quantum Lie algebras.

As a preliminary step towards physical applications it is necessary to identify the natural quantum generalizations of Lie algebras and to study their properties. Quantum generalizations $U_h(\mathfrak{g})$ of the enveloping algebras $U(\mathfrak{g})$ of Lie algebras \mathfrak{g} have been known since the work of Drinfeld [2] and Jimbo [3] and they have been found to play a central role in quantum integrable models. This has lead us in [4] to define quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ as certain submodules of $U_h(\mathfrak{g})$, modelling the way in which ordinary Lie algebras are naturally embedded in $U(\mathfrak{g})$.

Explicit examples of quantum Lie algebras were constructed in [4] using symbolic computer calculations, in particular for $\mathfrak{L}_h(\mathfrak{sl}_3)$, $\mathfrak{L}_h(\mathfrak{sl}_4)$, $\mathfrak{L}_h(\mathfrak{sp}_4)$ and $\mathfrak{L}_h(G_2)$. It was found empirically that in these quantum Lie algebras the quantum Lie products satisfy an intriguing generalization of the classical antisymmetry property. They are q -antisymmetric. This can be exhibited already in the simple example of $\mathfrak{L}_h(\mathfrak{sl}_2)$. This quantum Lie algebra is spanned by three generators X_h^+ , X_h^- and H_h with the quantum Lie product relations

$$\begin{aligned} [X_h^+, X_h^-]_h &= H_h, & [X_h^-, X_h^+]_h &= -H_h, \\ [H_h, X_h^\pm]_h &= \pm 2q^{\pm 1} X_h^\pm, & [X_h^\pm, H_h]_h &= \mp 2q^{\mp 1} X_h^\pm \\ [H_h, H_h]_h &= 2(q - q^{-1})H_h, & [X_h^\pm, X_h^\pm]_h &= 0. \end{aligned} \quad (1.1)$$

Here $q = e^h$ is the quantum parameter. Clearly for $q = 1$ the above reduces to the ordinary \mathfrak{sl}_2 Lie algebra. For $q \neq 1$ the Lie product is antisymmetric if the interchange of the factors is accompanied by $q \rightarrow q^{-1}$.

To convincingly establish that the quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ defined in [4] are the natural quantum generalizations of Lie algebras, three questions in particular should be answered:

1. Do the $\mathfrak{L}_h(\mathfrak{g})$ exist for all \mathfrak{g} ?

2. Are all $\mathfrak{L}_h(\mathfrak{g})$ associated to the same \mathfrak{g} isomorphic?
3. Do all $\mathfrak{L}_h(\mathfrak{g})$ have q -antisymmetric quantum Lie products?

These questions will be answered in the affirmative in this paper.

The paper is organized as follows: Section 2 contains preliminaries about quantized enveloping algebras $U_h(\mathfrak{g})$ and defines the concept of q -conjugation. In Section 3 we give a new definition of quantum Lie algebras \mathfrak{g}_h which is independent of any realization as submodules of $U_h(\mathfrak{g})$. We study the properties of the \mathfrak{g}_h . In Section 4 we recall the definition of the quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ and then show that all $\mathfrak{L}_h(\mathfrak{g})$ are isomorphic to \mathfrak{g}_h . It is in this way that we arrive in Theorem 1 at the answers to questions 2) and 3) above. In Section 5 we describe a construction for quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ for any finite-dimensional simple complex Lie algebra \mathfrak{g} , thus establishing their existence.

There are many natural questions about quantum Lie algebras which we do not address in this paper. These are question of representations, of the enveloping algebras, of exponentiation to quantum groups, of applications to physics and many more which we hope will be addressed in the future.

We do not wish to reserve the term *quantum Lie algebra* only for the particular algebras defined in this paper. Rather we view the algebras \mathfrak{g}_h and $\mathfrak{L}_h(\mathfrak{g})$ which are defined in Definitions 3.1 and 4.1 in terms of $U_h(\mathfrak{g})$ as particular examples of a more general concept of quantum Lie algebras. What a quantum Lie algebra should be in general is not yet known, i.e., there are not yet any satisfactory axioms for quantum Lie algebras. Finding such an axiomatic definition is an important problem. We hope that our study of the quantum Lie algebras arising from $U_h(\mathfrak{g})$ will help to provide the ideas needed to formulate the axioms. In particular we expect that the q -antisymmetry of the product discovered here will be an important ingredient.

There has been an important earlier approach to the subject of quantum Lie algebras. It was initiated by Woronowicz in his work on bicovariant differential calculi on quantum groups [5]. He defined a quantum Lie product on the dual space to the space of left-invariant one-forms. This has been developed further by several groups [6]. These quantum Lie algebras are n^2 -dimensional where n is the dimension of the defining representation of \mathfrak{g} and thus they do not have the same dimension as the classical Lie algebra except for $\mathfrak{g} = \mathfrak{gl}_n$. It has never been shown how to project them onto quantum Lie algebras of the correct dimension. Only recently Sudbery [7] has defined quantum Lie algebras for $\mathfrak{g} = \mathfrak{sl}_n$ which have the correct dimension $n^2 - 1$. These are isomorphic to our $(\mathfrak{sl}_n)_h(0)$ (set $s = -1, t = 0$ in Proposition 3.3). Schüler and Schmüdgen [8] have defined $n^2 - 1$ dimensional quantum Lie algebras for \mathfrak{sl}_n using left-covariant differential calculi. In [9] we explained

how our quantum Lie algebras lead to bicovariant differential calculi of the correct dimension.

Up to date information on quantum Lie algebras can be found on the World Wide Web at <http://www.mth.kcl.ac.uk/~delius/q-lie.html>

2 Preliminaries

We recall the definition of quantized enveloping algebras $U_h(\mathfrak{g})$ [2, 3, 10] in order to fix our notation. $U_h(\mathfrak{g})$ is an algebra over $\mathbb{C}[[h]]$, the ring of formal power series in an indeterminate h . In applications of quantum groups in physics, the parameter h does not need to be identified with Planck's constant. In general it will depend on a dimensionless combination of coupling constants and Planck's constant. We use the notation $q = e^h$.

The formal power series in h form only a ring, not a field. It is not possible to divide by an element of $\mathbb{C}[[h]]$ unless the power series contains a term of order h^0 . We will have to work with modules over this ring, rather than with vector spaces over a field as would be more familiar to physicists like ourselves. However $\mathbb{C}[[h]]$ is a principal ideal domain and thus many of the usual results of linear algebra continue to hold [11].

In the physics literature on quantum groups it is quite common to treat q not as an indeterminate but as a complex (or real) number. It is our opinion that in doing so, physicists lose much of the potential power of quantum groups. Keeping h as an indeterminate in the formalism will, when applied to quantum mechanical systems, lead to deeper insight.

Definition 2.1. Let \mathfrak{g} be a finite-dimensional simple complex Lie algebra with symmetrizable Cartan matrix a_{ij} . The *quantized enveloping algebra* $U_h(\mathfrak{g})$ is the unital associative algebra over $\mathbb{C}[[h]]$ (completed in the h -adic topology) with generators x_i^+ , x_i^- , h_i , $1 \leq i \leq \text{rank}(\mathfrak{g})$ and relations ¹

$$\begin{aligned} h_i h_j &= h_j h_i, & h_i x_j^\pm - x_j^\pm h_i &= \pm a_{ij} x_j^\pm, \\ x_i^+ x_j^- - x_j^- x_i^+ &= \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (x_i^\pm)^k x_j^\pm (x_i^\pm)^{1-a_{ij}-k} &= 0 \quad i \neq j. \end{aligned} \tag{2.1}$$

¹Our x_i^\pm are related to the X_i^\pm of [10] by $x_i^+ = q_i^{-h_i/2} X_i^+$ and $x_i^- = X_i^- q_i^{h_i/2}$ and it uses the opposite Hopf-algebra structure.

Here $\begin{bmatrix} a \\ b \end{bmatrix}_q$ are the q -binomial coefficients. We have defined $q_i = e^{d_i h}$ where d_i are the coprime integers such that $d_i a_{ij}$ is a symmetric matrix.

The Hopf algebra structure of $U_h(\mathfrak{g})$ is given by the comultiplication $\Delta : U_h(\mathfrak{g}) \rightarrow U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$ ($\hat{\otimes}$ denotes the tensor product over $\mathbb{C}[[h]]$, completed in the h -adic topology when necessary) defined by ²

$$\Delta(h_i) = h_i \hat{\otimes} 1 + 1 \hat{\otimes} h_i, \quad (2.2)$$

$$\Delta(x_i^\pm) = x_i^\pm \hat{\otimes} q_i^{-h_i/2} + q_i^{h_i/2} \hat{\otimes} x_i^\pm, \quad (2.3)$$

and the antipode S and counit ϵ defined by

$$S(h_i) = -h_i, \quad S(x_i^\pm) = -q_i^{\mp 1} x_i^\pm, \quad \epsilon(h_i) = \epsilon(x_i^\pm) = 0. \quad (2.4)$$

$U_h(\mathfrak{g})$ is quasitriangular with universal R -matrix $R \in U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$. The adjoint action of $U_h(\mathfrak{g})$ on itself is given, using Sweedler's notation [12], by

$$(\text{ad } x) y = \sum x_{(1)} y S(x_{(2)}), \quad x, y \in U_h(\mathfrak{g}). \quad (2.5)$$

If the Dynkin diagram of \mathfrak{g} has a symmetry τ which maps node i into node $\tau(i)$ then $U_h(\mathfrak{g})$ has a Hopf-algebra automorphism defined by $\tau(x_i^\pm) = x_{\tau(i)}^\pm$, $\tau(h_i) = h_{\tau(i)}$. Such τ are referred to as diagram automorphisms and except for rescalings of the x_i^\pm they are the only Hopf-algebra automorphisms of $U_h(\mathfrak{g})$.

Proposition 2.1 (Drinfel'd [13]). *There exists an algebra isomorphism $\varphi : U_h(\mathfrak{g}) \rightarrow U(\mathfrak{g})[[h]]$ such that $\varphi \equiv \text{id} \pmod{h}$ and $\varphi(h_i) = h_i$.*

Note. This is not a Hopf-algebra isomorphism however.

Proposition 2.2. *By (V^μ, π^μ) denote the $U(\mathfrak{g})$ -representation with highest weight μ , carrier space V^μ and representation map π^μ . Let $\{(V^\mu, \pi^\mu)\}_{\mu \in D_+}$ be the set of all finite-dimensional irreducible representations of $U(\mathfrak{g})$. D_+ is the set of dominant weights. Let $m_\lambda^{\mu\nu}$ denote the multiplicities in the decomposition of tensor product representations into irreducible $U(\mathfrak{g})$ representations*

$$V^\mu \otimes V^\nu = \bigoplus_{\lambda \in D_+} m_\lambda^{\mu\nu} V^\lambda. \quad (2.6)$$

Then

²Interchanging q and q^{-1} gives an alternative Hopf algebra structure, which is the one chosen in [4, 10].

1. $\{(V^\mu[[h]], \pi^\mu \circ \varphi)\}_{\mu \in D_+}$ is the set of all indecomposable representations of $U_h(\mathfrak{g})$ which are finite-dimensional, i.e., topologically free and of finite rank. Here φ is the isomorphism of Proposition 2.1.
2. The decomposition of $U_h(\mathfrak{g})$ tensor product representations into indecomposable $U_h(\mathfrak{g})$ representations is described by the classical multiplicities $m_\lambda^{\mu\nu}$

$$V^\mu[[h]] \hat{\otimes} V^\nu[[h]] = \bigoplus_{\lambda \in D_+} m_\lambda^{\mu\nu} V^\lambda[[h]]. \quad (2.7)$$

Proof. 1. is from Drinfel'd [13]. It follows immediately from the isomorphism property of φ and from the fact that the finite dimensional representations of $U(\mathfrak{g})$ have no non-trivial deformations. 2. the decomposition can be achieved by the same method as classically. A careful analysis shows that working over $\mathbb{C}[[h]]$ does not lead to complications. The reason is that all expressions appearing have a non-vanishing classical term. \square

Note. The $U_h(\mathfrak{g})$ modules $V[[h]]$ are not irreducible. Their submodules are of the form $cV[[h]]$ with $c \in \mathbb{C}[[h]]$ not invertible. In this setting Schur's lemma takes the following form:

Lemma 2.1 (Schur's lemma). *Let $V[[h]]$ and $W[[h]]$ be two finite-dimensional indecomposable $U_h(\mathfrak{g})$ -modules and let $f : V[[h]] \rightarrow W[[h]]$ be a $U_h(\mathfrak{g})$ -module homomorphism. Then if $f \neq 0$ then $f = cg$ with $c \in \mathbb{C}[[h]]$ and g an isomorphism.*

A central concept in the theory of quantum Lie algebras [4] is q -conjugation which in $\mathbb{C}[[h]]$ maps $h \mapsto -h$, i.e. $q \mapsto q^{-1}$.

Definition 2.2.

- (i) q -conjugation $\sim : \mathbb{C}[[h]] \rightarrow \mathbb{C}[[h]]$, $a \mapsto \tilde{a}$ is the \mathbb{C} -linear ring automorphism defined by $\tilde{h} = -h$.
- (ii) Let M, N be $\mathbb{C}[[h]]$ -modules. An additive map $\phi : M \rightarrow N$ is said to be q -linear if $\phi(\lambda a) = \tilde{\lambda} \phi(a)$, $\forall a \in M, \lambda \in \mathbb{C}[[h]]$.
- (iii) A q -conjugation on a $\mathbb{C}[[h]]$ module M is a q -linear involutive map ${}^\nabla : M \rightarrow M$ with ${}^\nabla = \text{id} \pmod{h}$.

Note the analogy between the concepts of q -conjugation and complex conjugation and between q -linear maps and anti-linear maps.

Remark. If M is a finite-dimensional $\mathbb{C}[[h]]$ -module then a q -conjugation $^\nabla$ on M is uniquely specified by giving a basis $\{b_i\}$ which is invariant. Then the q -conjugation takes the form $(\sum_i \lambda_i b_i)^\nabla = \sum_i \tilde{\lambda}_i b_i$. Conversely, for any q -conjugation on M there exists an invariant basis. It can be constructed from an arbitrary basis by adding correction terms order by order in h .

The unique q -linear algebra automorphism $\sim: U_h(\mathfrak{g}) \rightarrow U_h(\mathfrak{g})$ which extends q -conjugation on $\mathbb{C}[[h]]$ by acting as the identity on the generators x_i^\pm and h_i is a q -conjugation on $U_h(\mathfrak{g})$. It exists because the relations (1) are invariant under $q \mapsto q^{-1}$. We choose the isomorphism φ in Proposition 2.1 such that $\sim \circ \varphi = \varphi \circ \sim$. This q -conjugation is a coalgebra q -antiautomorphism of $U_h(\mathfrak{g})$, i.e., $\epsilon \circ \sim = \sim \circ \epsilon$, $\Delta \circ \sim = \sim \circ \Delta^T$ and it satisfies $S \circ \sim = \sim \circ S^{-1}$. The map \sim was introduced already in [13].

If in physical applications q were identified with a combination of a coupling constant and Planck's constant, then q -conjugation would correspond to the strong-weak coupling duality³. It has been observed in several quantum field theories, that such a duality transformation can form a symmetry of the theory. Affine Toda field theories in two dimensions [1] as well as supersymmetric Yang-Mills theory in four dimensions provide examples of this phenomenon. It is thus very desirable to have an algebraic structure, in which q -conjugation is incorporated. We hope that the study of this structure will one day enhance our understanding of the origin of strong-weak coupling duality in physics.

3 Quantum Lie algebras \mathfrak{g}_h

The quantized enveloping algebra $U_h(\mathfrak{g})$ is an infinite dimensional algebra. It is our aim to associate to it in a natural way a finite dimensional algebra which would be the quantum analog of the Lie algebra. Here our approach is based on the observation that classically a Lie algebra \mathfrak{g} is also the carrier space of the adjoint representation $\text{ad}^{(0)}$ of $U(\mathfrak{g})$. The superscript 0 is to remind us that this is the classical adjoint representation. It is defined by $(\text{ad}^{(0)} a) b = [a, b] \forall a, b \in \mathfrak{g}$. It follows from the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \quad \forall a, b, c \in \mathfrak{g} \quad (3.1)$$

that

$$(\text{ad}^{(0)} x) \circ [,] = [,] \circ (\text{ad}_2^{(0)} x), \quad \forall x \in U(\mathfrak{g}), \quad (3.2)$$

³ In some applications of quantum groups the relation between q and the coupling constant is not linear but exponential and then q -conjugation is not related to strong-weak duality

where $(\text{ad}_2^{(0)} x) = (\text{ad}^{(0)} \otimes \text{ad}^{(0)}) \Delta(x)$ is the tensor product representation carried by $\mathfrak{g} \otimes \mathfrak{g}$. Equation (3.2) states that the Lie product $[\cdot, \cdot]$ of \mathfrak{g} is a $U(\mathfrak{g})$ -module homomorphism from $\mathfrak{g} \otimes \mathfrak{g}$ to \mathfrak{g} .

Because of Proposition 2.2 we know that $\mathfrak{g}[[h]]$ is an indecomposable module of $U_h(\mathfrak{g})$. Let us denote the representation of $U_h(\mathfrak{g})$ on $\mathfrak{g}[[h]]$ by $\text{ad}^{(h)}$. Note that at this point there is no relation between the representation $\text{ad}^{(h)}$ of $U_h(\mathfrak{g})$ on $\mathfrak{g}[[h]]$ and the adjoint action ad of $U_h(\mathfrak{g})$ on $U_h(\mathfrak{g})$ defined in (2.5). Generalizing the above classical observation we obtain a natural definition for a quantum Lie algebra ⁴.

Definition 3.1. Let $[\cdot, \cdot]_h : \mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]] \rightarrow \mathfrak{g}[[h]]$ be a $U_h(\mathfrak{g})$ -module homomorphism which satisfies $[\cdot, \cdot]_h = [\cdot, \cdot] \pmod{h}$. $[\cdot, \cdot]_h$ gives $\mathfrak{g}[[h]]$ the structure of a non-associative algebra over $\mathbb{C}[[h]]$. We call this algebra $\mathfrak{g}_h = (\mathfrak{g}[[h]], [\cdot, \cdot]_h)$ a *quantum Lie algebra* and the product $[\cdot, \cdot]_h$ a *quantum Lie product*.

For each Lie algebra \mathfrak{g} this definition potentially gives many different quantum Lie algebras \mathfrak{g}_h , one for each choice of the homomorphism $[\cdot, \cdot]_h$. This would be unsatisfactory were it not for the fact that such a $U_h(\mathfrak{g})$ -module homomorphism is almost unique.

Proposition 3.1. *For a given $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ the quantum Lie algebra \mathfrak{g}_h is unique (up to a rescaling of the product by an invertible element of $\mathbb{C}[[h]]$). For $\mathfrak{g} = \mathfrak{sl}_n$ with $n > 2$ there is a family of quantum Lie algebras $(\mathfrak{sl}_n)_h(\chi)$ depending on a parameter $\chi \in \mathbb{C}((h))$ (see Proposition 3.3).*

Proof. The idea of the proof is simple: For $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ the adjoint representation appears in the tensor product of two adjoint representations with unit multiplicity. This is an empirical fact. Thus the homomorphism $[\cdot, \cdot]_h$ from $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$ into $\mathfrak{g}[[h]]$ with the requirement that $[\cdot, \cdot]_h \pmod{h} = [\cdot, \cdot]$ is unique by the weak form of Schur's lemma.

In the case $\mathfrak{g} = \mathfrak{sl}_n$ with $n > 2$ however, the adjoint representation appears with multiplicity two in the tensor product. Any module arising from a linear combination of the highest weight vectors of two adjoint modules is also an adjoint module and this leads to a one-parameter family of non-isomorphic weak quantum Lie algebras $(\mathfrak{sl}_n)_h(\chi)$.

We find it helpful to be more explicit here than necessary and to explain how the homomorphism $[\cdot, \cdot]_h$ is obtained from inverse Clebsch-Gordan coefficients. We begin with $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ and with the classical situation.

⁴As Ding has informed us, he and Frenkel have been pursuing similar ideas for some time. See also their paper [14] in which the utility of defining algebraic structures using $U_h(\mathfrak{g})$ module homomorphisms is stressed.

Let $\{v_a\}$ be a basis for \mathfrak{g} which contains a highest weight vector v_0 , i.e.,

$$(\text{ad}^{(0)} x_i^+) v_0 = 0, \quad (\text{ad}^{(0)} h_i) v_0 = \psi(h_i) v_0, \quad \forall i, \quad (3.3)$$

where ψ is the highest root of \mathfrak{g} . Let $P_a(x^-)$ be the polynomials in the x_i^- such that $v_a = (\text{ad}^{(0)} P_a(x^-)) v_0$. The adjoint representation matrices π in this basis are defined by

$$(\text{ad}^{(0)} x) v_a = v_b \pi_a^b(x). \quad (3.4)$$

In this paper we use the summation convention according to which repeated indices are summed over their range.

$\mathfrak{g} \otimes \mathfrak{g}$ contains a highest weight state \hat{v}_0 such that

$$(\text{ad}_2^{(0)} x_i^+) \hat{v}_0 = 0, \quad (\text{ad}_2^{(0)} h_i) \hat{v}_0 = \psi(h_i) \hat{v}_0, \quad \forall i, \quad (3.5)$$

For $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ this state is unique up to rescaling. The vectors

$$\hat{v}_a = (\text{ad}_2^{(0)} P_a(x^-)) \hat{v}_0 = K_a^{bc} v_b \otimes v_c \quad (3.6)$$

form a basis for \mathfrak{g} inside $\mathfrak{g} \otimes \mathfrak{g}$ such that

$$(\text{ad}_2^{(0)} x) \hat{v}_a = \hat{v}_b \pi_a^b(x) \quad (3.7)$$

with the same representation matrices π as in (3.4). Thus the map

$$\beta : v_a \mapsto \hat{v}_a = K_a^{bc} v_b \otimes v_c \quad (3.8)$$

is a $U(\mathfrak{g})$ -module homomorphism $\beta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$. The coefficients K_a^{bc} are called the Clebsch-Gordan coefficients. \mathfrak{g} and $\text{Im}(\beta)$ are irreducible modules and thus by Schur's lemma the homomorphism β is invertible on its image. Define $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ to be zero on the module complement of the image of β and on the image of β define $[\cdot, \cdot] = \beta^{-1}$. Then $[\cdot, \cdot]$ is the $U(\mathfrak{g})$ homomorphism from $\mathfrak{g} \otimes \mathfrak{g}$ to \mathfrak{g} , unique up to rescaling. It is the Lie product of \mathfrak{g} . On the basis it is given by

$$[v_a, v_b] = f_{ab}^c v_c, \quad \text{where } K_a^{bc} f_{bc}^d = \delta_a^d. \quad (3.9)$$

Thus the structure constants are given by the inverse Clebsch-Gordan coefficients.

We turn to the quantum case. Let \hat{v}_0 be a highest weight state inside $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$ satisfying the analog of (3.5)

$$(\text{ad}_2^{(h)} x_i^+) \hat{v}_0 = 0, \quad (\text{ad}_2^{(h)} h_i) \hat{v}_0 = \psi(h_i) \hat{v}_0, \quad \forall i, \quad (3.10)$$

where $\text{ad}^{(h)}$ is the deformed adjoint representation $\text{ad}^{(h)} = \text{ad}^{(0)} \circ \varphi$ and $\hat{v}_0 \pmod{h} \neq 0$. \hat{v}_0 generates the $U_h(\mathfrak{g})$ module $\mathfrak{g}[[h]]$ inside $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$. \hat{v}_0 must be unique up to rescaling, otherwise $\mathfrak{g}[[h]]$ would appear with multiplicity greater than one in $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$. We construct a basis $\{\hat{v}_a\}$ as in (3.6) using $P_a(x^-) \in U_h(\mathfrak{g})$ with the same polynomials P_a as in (3.6). This leads to quantum Clebsch-Gordan coefficients $K_a^{bc}(h) \in \mathbb{C}[[h]]$. We obtain a $U_h(\mathfrak{g})$ -module homomorphism $\beta : \mathfrak{g}[[h]] \rightarrow \mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$ as in (3.8).

β is invertible by the weak form of Schur's lemma. A homomorphism $[\cdot, \cdot]_h : \mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]] \rightarrow \mathfrak{g}[[h]]$ is obtained as above (3.9)

$$[v_a, v_b]_h = f_{ab}^c(h) v_c, \quad \text{where } K_a^{bc}(h) f_{bc}^d(h) = \delta_a^d. \quad (3.11)$$

Up to rescaling it is the unique such homomorphism with the property that $[\cdot, \cdot]_h \pmod{h} \neq 0$.

We now turn to $\mathfrak{g} = \mathfrak{sl}_n$ with $n > 2$ and again begin by considering the classical situation. There are two linearly independent highest weight vectors $\hat{v}_0^{(+)}$ and $\hat{v}_0^{(-)}$ in $\mathfrak{g} \otimes \mathfrak{g}$ which satisfy (3.5). They can be chosen so that

$$\sigma \hat{v}_0^{(\pm)} = \pm \hat{v}_0^{(\pm)}, \quad (3.12)$$

where σ is the bilinear map acting as $\sigma(v_a \otimes v_b) = v_b \otimes v_a$. Expressed differently, the Clebsch-Gordan coefficients $K_a^{(\pm)bc}$ defined as in (3.6) satisfy $K_a^{(\pm)bc} = \pm K_a^{(\pm)cb}$. Any linear combination of $\hat{v}_0^{(+)}$ and $\hat{v}_0^{(-)}$ is a highest weight state and leads to a homomorphism as described above but clearly only $\hat{v}_0^{(-)}$ leads to an *antisymmetric* Lie product.

In the quantum case too there are two linearly independent highest weight states satisfying (3.10). We can choose any linear combination and thus have a one-parameter family of $\hat{v}_0(\chi) = K_0^{bc}(\chi, h) (v_b \hat{\otimes} v_c)$. We impose $\hat{v}_0(\chi) \pmod{h} \neq 0$ as before. In this way we obtain the family $(\mathfrak{sl}_n)_h(\chi)$ of quantum Lie algebras. We will give these explicitly in Proposition 3.3. Certain values for χ will lead to a q -antisymmetric quantum Lie product (see Proposition 3.5). \square

Some important properties of \mathfrak{g} carry over immediately to \mathfrak{g}_h . Define root subspaces $\mathfrak{g}^{(\alpha)}$ of \mathfrak{g} by

$$\mathfrak{g}^{(\alpha)} = \{x \in \mathfrak{g} | (\text{ad}^{(0)} h_i) x = \alpha(h_i) x \ \forall i\}. \quad (3.13)$$

\mathfrak{g} possesses a gradation

$$\mathfrak{g} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}^{(\alpha)}, \quad [\mathfrak{g}^{(\alpha)}, \mathfrak{g}^{(\beta)}] \subset \mathfrak{g}^{(\alpha+\beta)}, \quad (3.14)$$

where R is the set of non-zero roots of \mathfrak{g} .

Proposition 3.2. *A quantum Lie algebra \mathfrak{g}_h possesses a gradation*

$$\mathfrak{g}_h = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}^{(\alpha)}[[h]], \quad [\mathfrak{g}^{(\alpha)}[[h]], \mathfrak{g}^{(\beta)}[[h]]_h \subset \mathfrak{g}^{(\alpha+\beta)}[[h]]. \quad (3.15)$$

Proof. According to Proposition 2.1 the algebra isomorphism $\varphi : U_h(\mathfrak{g}) \rightarrow U(\mathfrak{g})[[h]]$ leaves the h_i invariant and thus

$$\mathfrak{g}^{(\alpha)}[[h]] = \{x \in \mathfrak{g}[[h]] | (\text{ad}^{(h)} h_i) x = \alpha(h_i) x \ \forall i\}. \quad (3.16)$$

Let $X_\alpha \in \mathfrak{g}^{(\alpha)}[[h]]$ and $X_\beta \in \mathfrak{g}^{(\beta)}[[h]]$. From the homomorphism property of $[\cdot, \cdot]_h$ and the coproduct $\Delta(h_i) = h_i \hat{\otimes} 1 + 1 \hat{\otimes} h_i$ it follows that

$$\begin{aligned} (\text{ad}^{(h)} h_i) [X_\alpha, X_\beta]_h &= [(\text{ad}^{(h)} h_i) X_\alpha, X_\beta]_h + [X_\alpha, (\text{ad}^{(h)} h_i) X_\beta]_h \\ &= (\alpha(h_i) + \beta(h_i)) [X_\alpha, X_\beta]_h \end{aligned} \quad (3.17)$$

and thus $[X_\alpha, X_\beta]_h \in \mathfrak{g}^{(\alpha+\beta)}[[h]]$. \square

Choosing basis vectors $X_\alpha \in \mathfrak{g}^{(\alpha)}$ and $H_i \in \mathfrak{g}^{(0)}$ Proposition 3.2 implies that the quantum Lie product relations are of the form

$$\begin{aligned} [H_i, X_\alpha]_h &= l_\alpha(H_i) X_\alpha, & [X_\alpha, H_i]_h &= -r_\alpha(H_i) X_\alpha, \\ [H_i, H_j]_h &= f_{ij}^k H_k, & [X_\alpha, X_{-\alpha}]_h &= g_\alpha^k H_k, \\ [X_\alpha, X_\beta]_h &= N_{\alpha\beta} X_{\alpha+\beta} & \text{for } \alpha + \beta \in R, & \quad 0 \text{ otherwise.} \end{aligned} \quad (3.18)$$

This is similar in form to the Lie product relations of ordinary Lie algebras. The most important differences are

1. The structure constants are now elements of $\mathbb{C}[[h]]$, i.e., they depend explicitly on the quantum parameter.
2. $[H_i, H_j]_h$ does not have to be zero. Thus the grade zero subalgebra $\mathfrak{g}^{(0)}[[h]]$ of \mathfrak{g}_h is not abelian. We will nevertheless refer to it as the quantum Cartan subalgebra.
3. Each classical root α splits up into a “left” root l_α and a “right” root r_α . Classically they are forced to be equal because of the antisymmetry of the Lie product.

The quantum Clebsch-Gordan coefficients which describe the homomorphism $[\cdot, \cdot]_h : \mathfrak{g}_h \hat{\otimes} \mathfrak{g}_h \rightarrow \mathfrak{g}_h$ can be calculated directly by decomposing the tensor product representation. This is however very tedious in general. In [15] it was done for $(\mathfrak{sl}_n)_h$ in an indirect way by using the R-matrix of $U_q(\mathfrak{sl}_n)$. The method is based on realizing the quantum Lie algebra as a particular submodule of $U_h(\mathfrak{g})$ as explained in Section 4. The particular submodule used in [15] gives the quantum Lie algebra $(\mathfrak{sl}_n)_h(\chi = 1)$ but the method can be extended and gives the following result.

Proposition 3.3. *The parameter $\chi \in \mathbb{C}((h))$ of $(\mathfrak{sl}_n)_h(\chi)$ is a fraction $\chi = t/s$ with $s, t \in \mathbb{C}[[h]]$ and with the restriction that $(s+t)^{-1} \in \mathbb{C}[[h]]$. The Lie product relations for $(\mathfrak{sl}_n)_h(\chi)$ are*

$$\begin{aligned} [H_k, X_{ij}]_h &= l_{ij}(H_k) X_{ij}, & [X_{ij}, H_k]_h &= -r_{ij}(H_k) X_{ij}, \\ [H_i, H_j]_h &= f_{ij}^k H_k, & [X_{ij}, X_{ji}]_h &= g_{ij}^k H_k, \\ [X_{ij}, X_{kl}]_h &= \delta_{jk} \delta_{i \neq l} N_{ijl} X_{il} - \delta_{il} \delta_{j \neq k} M_{kij} X_{kj}, \end{aligned} \quad (3.19)$$

where $\{X_{ij}\}_{i,j=1 \dots n} \cup \{H_i\}_{i=1 \dots n-1}$ is a basis and the structure constants are explicitly given by

$$\begin{aligned} l_{ij}(H_k) &= (q^{1-k} \delta_{ki} - q^{-1-k} \delta_{k,i-1})(s + t q^n) \\ &\quad - (q^{k-1} \delta_{kj} - q^{k+1} \delta_{k,j-1})(s + t q^{-n}), \end{aligned} \quad (3.20)$$

$$r_{ij}(H_k) = -l_{ji}(H_k), \quad (3.21)$$

$$\begin{aligned} f_{ij}^k &= \delta_{ij} \left(s (q^{k+1} - q^{-k-1}) + t (q^{n+1-i} - q^{-n-1+i}) \right) \\ &\quad + s \delta_{k < i} (q + q^{-1})(q^k - q^{-k}) \\ &\quad + t \delta_{k > i} (q + q^{-1})(q^{n-k} - q^{-n+k}) \\ &\quad + \delta_{i,j-1} \left(s \delta_{k \leq i} (q^{-k} - q^k) + t \delta_{k > i} (q^{k-n} - q^{-k+n}) \right) \\ &\quad + \delta_{j,i-1} \left(s \delta_{k \leq j} (q^{-k} - q^k) + t \delta_{k > j} (q^{k-n} - q^{-k+n}) \right), \end{aligned} \quad (3.22)$$

$$\begin{aligned} g_{ij}^k &= q^{i-j} \left(s (q^k \delta_{k < j} - q^{-k} \delta_{k < i}) + t (q^{n-k} \delta_{k \geq i} - q^{k-n} \delta_{k \geq j}) \right) \\ N_{ijl} &= q^{1/2-j} (s + t q^n), & M_{kij} &= q^{i-1/2} (s + t q^{-n}) \end{aligned} \quad (3.23)$$

(We use a generalized Kronecker delta notation, e.g., $\delta_{i \leq j} = 1$ if $i \leq j$, 0 otherwise.)

The restriction that if χ is written as $\chi = t/s$ then $s+t$ has to be invertible comes from the requirement that the quantum Lie product should not vanish modulo h . For details of the calculation leading to the above formulae we refer the reader to [15].

The Lie algebra \mathfrak{sl}_n with $n > 2$ possesses an automorphism which is due to the symmetry of the Dynkin diagram. It would be natural to require that this automorphism survives also at the quantum level. By inspecting the above Lie product relations we find

Proposition 3.4. *The quantum Lie algebra $(\mathfrak{sl}_n)_h(\chi)$ possesses the Dynkin diagram automorphism*

$$\tau(X_{ij}) = -X_{n+1-j, n+1-i}, \quad \tau(H_i) = H_{n-i} \quad (3.24)$$

iff $\chi = 1$.

This is the reason why in [15] we focused our attention on the case of $\chi = 1$.

The most basic property of a Lie product is its antisymmetry. In quantum Lie algebras this has found an interesting generalization.

Proposition 3.5. *The quantum Lie product of \mathfrak{g}_h for $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ and of $(\mathfrak{sl}_n)_h(\chi)$ with $\tilde{\chi} = \chi$ is q -antisymmetric, i.e., there exists a q -conjugation ${}^\nabla : \mathfrak{g}_h \rightarrow \mathfrak{g}_h$ consistent with the gradation (3.15) such that*

$$[a, b]_h^\nabla = -[b^\nabla, a^\nabla]_h. \quad (3.25)$$

Thus, choosing the basis in (3.18) so that $X_\alpha^\nabla = X_\alpha$, $H_i^\nabla = H_i$, the structure constants satisfy

$$r_\alpha = \tilde{l}_\alpha, \quad f_{ij}{}^k = -\tilde{f}_{ji}{}^k, \quad g_\alpha{}^k = -\tilde{g}_{-\alpha}{}^k, \quad N_{\alpha\beta} = -\tilde{N}_{\beta\alpha}. \quad (3.26)$$

Proof. For $(\mathfrak{sl}_n)_h$ the statement can be verified directly from the expressions in Proposition 3.3. For $\mathfrak{g} \neq \mathfrak{sl}_n$ we use the same notation as in the proof of Proposition 3.1. The adjoint representation appears with multiplicity one in the tensor product and thus we know that the highest weight state $\hat{v}_0 = K_0^{ab}(h) v_a \otimes v_b$ in $\mathfrak{g}[[h]] \hat{\otimes} \mathfrak{g}[[h]]$ satisfying (3.10) is unique up to rescaling. $\tilde{v}_0^T = K_0^{ba}(-h) v_a \otimes v_b$ also satisfies the highest weight condition (3.10).

$$\begin{aligned} (\text{ad}_2^{(h)} x_i^+) \tilde{v}_0^T &= ((\text{ad}^{(h)} \otimes \text{ad}^{(h)}) \Delta(x_i^+)) \tilde{v}_0^T \\ &= \sim [((\text{ad}^{(h)} \otimes \text{ad}^{(h)}) \Delta^T(x_i^+)) \hat{v}_0^T] \\ &= \sim \left[(\text{ad}_2^{(h)} x_i^+) \hat{v}_0 \right]^T \\ &= 0. \end{aligned} \quad (3.27)$$

We used that $\sim \circ (\text{ad}^{(h)} x) = (\text{ad}^{(h)} \tilde{x}) \circ \sim$ (which follows from $\sim \circ \varphi = \varphi \circ \sim$), that $\tilde{v}_a = v_a$ and that $\sim \circ \Delta = \Delta^T \circ \sim$. Thus $\hat{v}_0' = \frac{1}{2}(\hat{v}_0 - \tilde{v}_0^T)$ is a highest weight state (proportional to \hat{v}_0 by uniqueness). It is non-zero because it is non-zero classically. Following a similar calculation to the above one finds that it leads to Clebsch-Gordan coefficients $K_a'^{bc}(h) = \frac{1}{2}(K_a^{bc}(h) - K_a^{cb}(-h))$. These are manifestly q -antisymmetric. Following through the construction of the structure constants one finds $f_{ab}'{}^c(h) = -f_{ba}'{}^c(-h)$. \square

4 Quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$

In Definition 3.1 quantum Lie algebras are defined abstractly, i.e., independently of any specific realization. In [4] quantum Lie algebras were defined as

concrete objects, namely as certain submodules of the quantized enveloping algebras $U_h(\mathfrak{g})$. This definition is based on the observation that an ordinary Lie algebra \mathfrak{g} can be naturally viewed as a subspace of its enveloping algebra $U(\mathfrak{g})$ with the Lie product on this subspace given by the adjoint action of $U(\mathfrak{g})$. Thus it is natural to define a quantum Lie algebra as an analogous submodule of the quantized enveloping algebra $U_h(\mathfrak{g})$ with the quantum Lie product given by the adjoint action of $U_h(\mathfrak{g})$. Before we can state the precise definition we need some preliminaries.

The Cartan involution $\theta : U_h(\mathfrak{g}) \rightarrow U_h(\mathfrak{g})$ is given by the same formulas as in the classical case: $\theta(x_i^\pm) = x_i^\mp$, $\theta(h_i) = -h_i$. It is an algebra automorphism and a coalgebra antiautomorphism, i.e., $\Delta \circ \theta = (\theta \hat{\otimes} \theta) \circ \Delta^T$ and $S \circ \theta = \theta \circ S^{-1}$. We define a tilded Cartan involution by composing the Cartan involution with q -conjugation, i.e., $\tilde{\theta} = \sim \circ \theta$. Similarly we define a tilded antipode as $\tilde{S} = \sim \circ S$. With respect to the adjoint action defined in (2.5) they satisfy $(\text{ad } \tilde{\theta}(a)) \tilde{\theta}(b) = \tilde{\theta}((\text{ad } a) b)$ and $(\text{ad } \tilde{S}(a)) \tilde{S}(b) = \tilde{S}((\text{ad } S^{-1}(a)) b)$ for all $a, b \in U_h(\mathfrak{g})$.

Definition 4.1. A quantum Lie algebra $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$ is a finite-dimensional indecomposable ad - submodule of $U_h(\mathfrak{g})$ endowed with the quantum Lie product $[a, b]_h = (\text{ad } a) b$ such that

1. $\mathfrak{L}_h(\mathfrak{g})$ is a deformation of \mathfrak{g} , i.e., there is an algebra isomorphism $\mathfrak{L}_h(\mathfrak{g}) \cong \mathfrak{g} \pmod{h}$.
2. $\mathfrak{L}_h(\mathfrak{g})$ is invariant under $\tilde{\theta}$, \tilde{S} and any diagram automorphism τ .

A weak quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$ is defined similarly but without the requirement 2.

The existence of a Cartan involution and an antipode on $\mathfrak{L}_h(\mathfrak{g})$ plays an important role in the investigations into the general structure of quantum Lie algebras in [4]. In particular it allows the definition of a quantum Killing form. The invariance under the diagram automorphisms τ is less important but is clearly a natural condition to impose. It is shown in [4] that given any weak quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$, one can always construct a true quantum Lie algebra $\mathfrak{L}_h(\mathfrak{g})$ which satisfies property 2 as well. Thus this extra requirement is not too strong.

We now come to the relation between the abstract quantum Lie algebras \mathfrak{g}_h of Definition 3.1 and the concrete weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ of Definition 4.1.

Proposition 4.1. All weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$ are isomorphic to the quantum Lie algebra \mathfrak{g}_h as algebras (or to $(\mathfrak{sl}_n)_h(\chi)$ for some χ in the case of $\mathfrak{g} = \mathfrak{sl}_n$).

Proof. By definition $\mathfrak{l}_h(\mathfrak{g})$ is a finite-dimensional, indecomposable $U_h(\mathfrak{g})$ module. Condition 1 of the definition implies that the representation of $U_h(\mathfrak{g})$ carried by this module is a deformation of the representation of $U(\mathfrak{g})$ carried by \mathfrak{g} . There is only one such deformation, namely the adjoint representation $\text{ad}^{(h)}$ carried by $\mathfrak{g}[[h]]$. Thus $\mathfrak{l}_h(\mathfrak{g})$ is isomorphic to $\mathfrak{g}[[h]]$ as a $U_h(\mathfrak{g})$ module. The identity

$$\sum (\text{ad}(\text{ad } x_{(1)}) a) ((\text{ad } x_{(2)}) b) = (\text{ad } x) ((\text{ad } a) b) \quad (4.1)$$

can be rewritten using that, when restricted to $\mathfrak{l}_h(\mathfrak{g}) \subset U_h(\mathfrak{g})$, $[a, b]_h = (\text{ad } a) b = (\text{ad}^{(h)} a) b$.

$$(\text{ad}^{(h)} x) \circ [,]_h = [,]_h \circ (\text{ad}_2^{(h)} x), \quad \forall x \in U(\mathfrak{g}). \quad (4.2)$$

This states that the quantum Lie product on $\mathfrak{l}_h(\mathfrak{g})$ is a $U_h(\mathfrak{g})$ -module homomorphism and thus is a quantum Lie product in the sense of Definition 3.1. \square

Remark. One should not confuse the adjoint *action* ad with the adjoint *representation* $\text{ad}^{(h)}$. The adjoint action ad is defined using the coproduct and the antipode as

$$(\text{ad } x) y = x_{(1)} y S(x_{(2)}) \quad \forall x, y \in U_h(\mathfrak{g}).$$

The adjoint representation $\text{ad}^{(h)}$ is defined using the algebra isomorphism $\varphi : U_h(\mathfrak{g}) \rightarrow U(\mathfrak{g})[[h]]$ of Proposition 2.1 as

$$(\text{ad}^{(h)} x) a = (\text{ad}^{(0)} \varphi(x)) a \quad \forall x \in U_h(\mathfrak{g}), a \in \mathfrak{g}[[h]].$$

Thus the adjoint action is determined by the h -deformed Hopf-algebra structure whereas the adjoint representation is determined by only the h -deformed algebra structure. From this point of view it is surprising that the two ever coincide. But the weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ are exactly those embeddings of $\mathfrak{g}[[h]]$ into $U_h(\mathfrak{g})$ on which ad and $\text{ad}^{(h)}$ coincide and we will establish their existence in the next section.

Proposition 4.1 allows us to answer two important questions about the concrete quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$ which were left unanswered in [4].

Theorem 1. *Given any finite-dimensional simple complex Lie algebra \mathfrak{g} .*

1. *All quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ are isomorphic as algebras.*

2. All quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ have q -antisymmetric Lie products.

Proof. 1. For $\mathfrak{g} \neq \mathfrak{sl}_{n>2}$ this is obvious from Proposition 4.1 and the uniqueness of \mathfrak{g}_h according to Proposition 3.1. For $\mathfrak{g} = \mathfrak{sl}_{n>2}$ the requirement of τ -invariance in Definition 4.1 implies through Proposition 3.4 that $\mathfrak{L}_h(\mathfrak{sl}_n)$ can be isomorphic only to $(\mathfrak{sl}_n)_h(\chi = 1)$. 2. This is obvious because \mathfrak{g}_h and $(\mathfrak{sl}_n)_h(\chi = 1)$ have q -antisymmetric Lie products according to Proposition 3.5. \square

5 Construction of quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$

There is a general method for the construction of weak quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ and quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$. The method was presented in [15] for $\mathfrak{g} = \mathfrak{sl}_n$ but it works for any finite-dimensional simple complex Lie algebra \mathfrak{g} as we will discuss here.

We begin with a lemma giving a construction of ad-submodules of $U_h(\mathfrak{g})$.

Lemma 5.1. *Let A be any element of $U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g})$ satisfying $A \Delta(x) = \Delta(x) A$, $\forall x \in U_h(\mathfrak{g})$. Let $V[[h]]$ be any finite-dimensional indecomposable $U_h(\mathfrak{g})$ module and let π_{ij} be the corresponding representation matrices. Then the elements*

$$A_{ij} = (\pi_{ij} \otimes \text{id}) A \in U_h(\mathfrak{g}) \quad (5.1)$$

span an ad-submodule of $U_h(\mathfrak{g})$ which is isomorphic to a submodule of $V[[h]]^ \hat{\otimes} V[[h]]$, i.e.,*

$$(\text{ad } x) A_{ij} = A_{kl} \pi_{ki}^*(x_{(1)}) \pi_{lj}(x_{(2)}), \quad \forall x \in U_h(\mathfrak{g}). \quad (5.2)$$

Here π^ denotes the dual (contragredient) representation to π defined by*

$$\pi_{ki}^*(x) = \pi_{ik}(S(x)). \quad (5.3)$$

Proof. We first calculate

$$\begin{aligned} x A_{ij} &= (\pi_{ij} \otimes \text{id}) (1 \otimes x) A \\ &= (\pi_{ij} \otimes \text{id}) (S(x_{(1)}) \otimes 1) A (x_{(2)} \otimes x_{(3)}) \\ &= \pi_{ik}(S(x_{(1)})) A_{kl} \pi_{lj}(x_{(2)}) x_{(3)}. \end{aligned} \quad (5.4)$$

Then, using (5.3)

$$\begin{aligned} (\text{ad } x) A_{ij} &= x_{(1)} A_{ij} S(x_{(2)}) \\ &= A_{kl} \pi_{ki}^*(x_{(1)}) \pi_{lj}(x_{(2)}) x_{(3)} S(x_{(4)}) \\ &= A_{kl} \pi_{ki}^*(x_{(1)}) \pi_{lj}(x_{(2)}) \end{aligned} \quad (5.5)$$

\square

This lemma can be applied to construct weak quantum Lie algebras.

Proposition 5.1. *Let $A = h^{-1}(R^T R - 1)$ where R is the universal R -matrix of $U_h(\mathfrak{g})$ and R^T the same with the tensor factors interchanged (i.e., if $R = \sum a_i \otimes b_i$ then $R^T = \sum b_i \otimes a_i$). Let $\{e_i\}$ be a basis for the $U_h(\mathfrak{g})$ module $V[[h]]$ and let π_{ij} be the corresponding representation matrices. Choose a basis $\{v_a\}$ for the adjoint representation $\mathfrak{g}[[h]]$ of $U_h(\mathfrak{g})$ and let $K : \mathfrak{g}[[h]] \rightarrow V[[h]]^* \hat{\otimes} V[[h]]$, $v_a \mapsto \hat{v}_a = K_a^{ij} (e_i^* \otimes e_j)$ be a $U_h(\mathfrak{g})$ -module homomorphism, i.e., the K_a^{ij} are quantum Clebsch-Gordan coefficients. Then the elements*

$$A_a = K_a^{ij} (\pi_{ij} \otimes id) \quad A \in U_h(\mathfrak{g}) \quad (5.6)$$

span a weak quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g}) = \text{span}_{\mathbb{C}[[h]]}\{A_a\}$.

Proof. The expression $A = h^{-1}(R^T R - 1)$ is well defined because $R = 1 \pmod{h}$. It follows from the defining property $R \Delta(x) = \Delta^T(x) R \quad \forall x \in U_h(\mathfrak{g})$ of the R -matrix that $A \Delta(x) = \Delta(x) A, \quad \forall x \in U_h(\mathfrak{g})$. It is then clear from Lemma 5.1 that the A_a span an ad-submodule of $U_h(\mathfrak{g})$. It follows from the definition of the Clebsch-Gordan coefficients K_a^{ij} that this ad-submodule is either isomorphic to the adjoint representation or zero. R satisfies $R = 1 + h r + \mathcal{O}(h^2)$ where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is the classical r -matrix. Thus $A = r + r^T \pmod{h} \in \mathfrak{g} \otimes \mathfrak{g}$ and $A_a \pmod{h} \in \mathfrak{g}$. It follows that $\text{span}_{\mathbb{C}[[h]]}\{A_a\} = \mathfrak{g} \pmod{h}$. \square

Using the fact, established in [4], that given a weak quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$ one can always construct a true quantum Lie algebra $\mathfrak{L}_h(\mathfrak{g})$, we arrive at the announced existence result.

Theorem 2. *For any finite-dimensional simple complex Lie algebra \mathfrak{g} there exists at least one quantum Lie algebra $\mathfrak{L}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$.*

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